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On Interrelationship of b – Lindelof & Second Countable Spaces As Well As Countably b – Compact & Sequentially b – Compactness

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ABSTRACT:
This paper is devoted to introduce and study b – Lindelof , Countably b – compact & sequentially b – compact topological spaces and their interrelationship . In this context the concept of second countable b – space is projected and interrelated to b – Lindelof space with proper examples.
The b – convergence of a sequence due to b – open sets in topological space has been conceptualized and the relation of b – convergence with→ b – continuity and b – irresolute mapping has been discovered here . It also deals with the relation between b – convergent sequence and convergence of a sequence in a space with a suitable example.

Keywords: b – continuity, b – convergent sequence, sequentially b – compact space, countably b – compact space, b – Lindelof space, second countable b – space.

§ 1. INTRODUCTION:

The notions of b – open sets have been introduced and investigated by D. Andrijevic [1] In 2007, M. Caldas & S . Jafari projected some applications of → b – open sets in topological spaces [4].

The class of generalized closed sets & regular generalized closed sets was coined & framed by N . Levine [6] and N. Palanniappan & K. Chandrasekhar Rao , respectively .
We here, introduce and study b – Lindelof space, countably b – compact space and sequentially b – compact space.
We also study the new concept of second countable b – space along with the b – converge of a sequence and its behavior under b – continuity / irresolute in a topological space .

As usual troughout this paper (X,T) means a topological spaces on which no separation axioms are assumed unless otherwise mentioned .
For a subset A of a space (X,T) , cl (A) and int (A) stand as the closure of A and the interior of A respectively.
Also, X – A or A^c represents the complement of A in X.

Now , the following definations are recalled which are useful in the sequel
Definition (1.1): A subset A of a space (X, T) is said to be b – open [1] if
$A \subset \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$.

Definition (1.2): A subset A of a space (X,T) is said to be regular closed if
A = cl (int(A))

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Definition (1.3) : A subset A of a space (X,T) is said to be generalized b – closed (briefly gb – closed) [5] set if

bcl (A) ⊂ U whenever A⊂U & U is open in X.

Definition (1.4) :A subset A of a space (X,T) is said to be
(i) generalized closed (briefly g – closed) [6] set if cl (A) ⊂ U whenever A ⊂ U & U is open in X .

(ii) A subset A of a space (X,T) is said to be generalized regular b – closed (briefly grb – closed) [3] if rcl (A) ⊂ U Whenever A ⊂ U & U is b – open in (X,T)
The compliments of the above mentioned closed sets are their respective open sets.
The intersection of all b – closed sets of X containing A is called →b – closure of A and is denoted by b – cl (A).
The union of all b – open sets of X contained in A is called the → b – interior of A and is denoted by b – int (A) .

The family of all b – open (respectively b – closed) sets of (X,T) is denoted by BO (X) (respectively BC (X)) . The family of b – open sets of (X,T) containing a point x ∈ X is denoted by BO (X,x) .

Now, the compactness is dealt with converging the sets by b – open sets as mentioned in the following definitions:

Definition (1.5) :
In a topological space (X,T) a collection C of b – open sets in X is called a b – open cover of A ⊆ x if A ⊆ U { V_r : V_r ∈ C } .

Definition (1.6) :
A topological space (X,T) is called a b – compact space / b – Lindelof space if every cover of X by b – open sets has a finite sub cover / countable sub cover .

Definition (1.7) :
In a topological space (X,T) , a subset A of X is said to be b – compact relative to X if for every b – open cover C of A , there is a finite sub collection C^* of C that covers A .

Definition (1.8):
A subspace of a topological space, which is b – compact as a topological space in its own right, is said to be b – compact subspace.

The following Lemma (1.1) is enunciated for above definitions to be consistent:
Lemma (1.1)
1) Every b – compact space is a b – Lindelof space.

2) Every b – Lindelof space is a Lindelof space.

3) Every countable space is a b – Lindelof space.
3) a) A b – Lindelof space need not be a b – compact space.

3) b – compactness is not hereditary

Proof: The statement follows from definitions (1.6), (1.7) & (1.8).

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§2. Second Countable b – Space:

Definition (2.1) : A topological space (X,T) is said to be a second countable→ b – Space or a second axiom b – space if it carries the following axiom, known as the "Second Axiom of b – Countability" (framed analogous to second axiom of countability) :

[C] There exists a countable b – open base for the topology T.

We, however, coin b – open base for the space (X,T) as a sub collection B ⊆ BO (X) such that every member of T is a union of members of B .

Thus, a topological space (X,T) is called a second countable b – space iff there exists a countable b – open base for T .

Theorem (2.1) :

Every second countable b – space is a b – Lindelof space.

Proof:

Let the topological space (X,T) be a second countable b – space .

Let $\{G_\alpha\}_{\alpha\in\Delta}$ be a b – open cover of X ,

then $X = \bigcup_{\alpha\in\Delta} G_\alpha$ - - - - - - - - - - - (1)

As X being second countable b – space , there exists a countable b – open base for the topology T. Let B = (V_n) be a countable b – open base for T. From (1) it follows that for each $x \in X$, there exists $\alpha_x \in \Delta$ such that

$x \in G_\alpha$ - - - - - - - - - - - - (2)

Now , since B is a b – open base for T , each open set is a union of some members of B . It therefore, follows from statement (2) that for each

$x \in X$, ∍ $Vn_x \in B$

such that $x \in Vn_x \subseteq Gr_x$ - - - - - - - - - - - - -(3)

Hence , $X = \bigcup_{x\in X} Vn_x$ - - - - - - - - - - - - - - (4)

Since , the family $\{Vn_x : x \in X\} \subseteq B$ and B is countable ,it follows that the family $\{ Vn_x : x \in X\}$ is countable Hence , We can write ,

$\{ Vn_x : x \in X\} = \{ Vn_x : k \in \Delta_o\}$ - - - - - - - - - - - - - (5)

Where Δ_o is a countable index set . This means that for each $k \in \Delta_o$,

∍ $x_k \in x$ such that $Vn_k = Vn_{xk}$

Hence , according to (2) & (3) , for each $k \in \Delta_o$, we select one index

$\alpha_{x_k} \in \Delta$

Such that $Vn_{xk} \subseteq Gr_{xk}$ - - - - - - - - - - - -(6)

Thus , from (4), (5), (6) , we have

$$X = \bigcup_{x \in X} Vn_x = \bigcup_{k \in \Delta} Vn_{xk} \subseteq \bigcup_{k \in \Delta o} {}^{G}\alpha_{xk}$$

But always $\bigcup_{k \in \Delta o} {}^{G}\alpha_{xk} \subseteq x$

Hence , $x = \bigcup_{k \in \Delta o} {}^{G}\alpha_{xk}$ - - - - - - - - - - - - -(7)

Moreover the family$\{Gr_{xk} : K \in \Delta_o\}$
is countable , hence by (7) , this family is a countable b-open subcovering of X.

Thus , every second countable b-space is a b-Lindeof space.

Hence, the theorem

§3. Sequentially b-compact spaces;

The notion of convergence is fundamental in analysis and topology. Before we take up the concept of sequentially b-compact spaces and countably b - compact spaces , we project the notion of b-convergence of a sequence, b-limit of a sequence , b-accumulation point of a set in a topological space in the following manner:

Definition (3.1) :

Let (X,T) be a topological space and $A \subseteq X$.

A point $p \in X$ is called a b-limit point (or a b-cluster point or a b-accumulation point) of A if every b-open set containing p contains a point of A other than p.

i.e. symbolically

$$[\, p \in (X,T) \wedge A \subseteq X \,] \Rightarrow [p = A \text{ b – limit point for } A]$$
$$\Leftrightarrow [\, \forall N \in Bo(X) \wedge_p \in X] \Rightarrow$$
$$[[N-\{p\}] \cap A \neq \phi \,]]$$

Definition (3.2) : b – convergent sequences :

A sequence $\{X_n\}$ in a topological space (X,T) is said to be b-convergent to a point X_0 or to converge to a point $x_o \in X$ with respect to b-open sets , written as $x_n \xrightarrow{\text{b-cgt}} x_o$, if for every b-open set L containing X_0 , there exists a positive integer m , s.t. $n \geqslant m \Rightarrow x_n \in L$.

This concept is symbolically presented as:

$$x_n \xrightarrow{\text{b-cgt}} x_o \Leftrightarrow \text{b-}\lim_{n \to \alpha} x_n = x_o$$

Obviously, a sequence $\{x_n\}$ in a topological space (X,T) is said to be b-convergent to a point x_o in X iff it is eventually in every b-open set containing x_o .

Definition (3.3) : b – limit point of a sequence:

A point X_0 in X is said to be b-limit point of a sequence $\{X_n\}$ in a topological space (X,T) iff every b – open set L containing x_0 there exists a +ve integer m such that $n \geqslant m \Rightarrow x_n \in L$.

This means that a sequence $\{x_n\}$ in a topological space (X,T) is said to have $x_o \in x$ as a b – limit point iff for every b – open set containing X_0 contains X_n for infinitely many n.

Definition (3.4): sequentially b- compact spaces:

A topological space (X, T) is said to be sequentially b- compact iff every sequence in x contains a sub sequence which is b- convergent to a point of X.

Definition (3.5) : Countably b-compact spaces:

A topological space (X,τ) is said to be countably b-compact (or to have b-Bolzano Weierstrass Property) if every infinite subset of X has at least one→ b-limit point in X

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A topological space (X,τ) is known as countably b-compact if every countable T-b-open cover of X has a finite sub-cover.

Remark (3.1):

(i) Every finite subspace of a topological space is sequentially b-compact.

(ii) Every b-compact space is a countably b-compact space.

(iii) Every cofinite topological space is a countably b-compact space.

Theorem (3.1):

Every sequentially b-compact topological space (X,τ) is countablly b-compact.

Proof:

Let (X,τ) be a sequentially b-compact topological space. Let E be any infinite subset of X. Then there exists an infinite sequence $\{x_n\}$ in E with distinct terms.

Since, (X,τ) is sequentially b-compact the sequence $\{x_n\}$ contains a sub sequence $\{x_{nk}\}$ which is b-convergent to $x_o \in X$.

This means that each b-open set containing x_O contains an infinite number of elements of E.

Hence, x_o is a b-accumulation point of E.

Thus, every infinite subset E of X has at least one b-accumulation point in X. Consequently (X,τ) is countably b-compact.

i.e, sequentially b-compactness implies countable b-compactness.

Hence, the theorem

Remark (3.2) :

A countably b-compact space is not necessary sequentially b-compact as illustrated by following example :

Example (3.1) :

N = { n : n is a natural number}.

Let T be topology on N generated by the family H ={ { 2n-1 , 2n} : N ∈ N of subsets of N.

Let E be a non empty subset of N .

Let $m_0 \in E$ if m_0 is even, then m_0-1 is a b-accumulation point of E. Hence, every non – empty subset of N has a b – accumulation point , so that (N,T) is countably b - comapct.

Also , (N,T) is not sequentially b – compact because the sequence {2n-1 :n∈N } has no b - convergent sub sequence.

Therefore

Countably b- compactness $\not\Rightarrow$ b – sequentially compactness.

$\not\Rightarrow$ b- compactness

Definition (3.6) : b- continuity at a point :

A mapping $f : (X,\tau) \longrightarrow (y,\sigma)$ from one topological space (X,τ) to another topological space (y,σ) is said to be b-continuous at a point $x_o \in x$ if for every σ - open set V containing $f(X_0)$ there exists a b- open set L in (x,τ) containing x_0 such that $f(L) \subseteq V$.

Definition (3.6) (a) : b – irresolute at a point ;

A mapping $f: (x,\tau) \to (y,\sigma)$ from one topological space (x,τ) to another topological space (y,σ) is said to be b-irresolute at a point $\rightarrow$ $x_0 \in X$ if for every b-open set V containing f (x_0) there exists a b- open set L in (x,τ) containing x_0 such that $f(L) \subseteq V$.

We, here, produce the following two theorems concerned with b-convergence and convergence of a sequence and its image sequence under b- continuity & b – irresoluteness:

Theorem (3.2):
In a topological space (X,T) if a sequence $\{x_n\}$ is b- convergent to a point $x_0 \in X$, then it is also simply convergent to the point. But the converse may not be true.

Proof:

Let K be an open set in a topological space (X, T) containing $\rightarrow x_0 \in X$, then K is also a b – open set .

Now, let $\{x_n\}$ be a b – convergent sequence which b-converges to the point $x_o \in x$. Then for every b – open set L containing x_o there exists a +ve integer m such that $x_n \in L$ for all $n \geqslant m$.

Thus, $x_n \xrightarrow{b\text{-}cgt} x_o \Leftrightarrow \forall L \in BO(X)$ and $x_o \in L$ implies that there exists a positive integer m > 0 such that $\forall n \geqslant m \Rightarrow x_n \in L$. This is also true for every open set $K \in T$ .Since K is an arbitrary open set containing x_o,

Hence, $x_n \xrightarrow{cgt} x_o$

But "the converse is not true" is supported by the following example:

Example (3.2):
Let X = {a,b,c},T= {ϕ,{a,b},x} then {b} is a b-open set but not an open set.Let x_n = a for all n, then $x_n \xrightarrow{cgt}$ a ,as well as $x_n \xrightarrow{cgt}$ b, because open subsets containing a and b are {a,b} and X.
But $\{x_n\}$ is not b-cgt to 'b' because there exists a b-open set containing 'b' as {b} which does not contain 'a' .

Hence the theorem,

Theorem (3.3):
If $f(X,\tau) \longrightarrow (Y,\sigma)$ be a b-continuous mapping from a topological space (X, τ) into another topological space (Y,σ) and $\{x_n\}$ be b-convergent to $\rightarrow f(x_o) \in Y$.

Proof:
Given that the mapping f: $(X,\tau) \rightarrow (Y, \sigma)$ is b-continuous so that it is b-continuous at every point of X.
Let $\{x_n\}$ be a sequence in (x,τ) which is b-convergent to$\rightarrow x_o \in x$.
Let V be a σ - open set in (Y, σ) containing $f(x_o)$. Then the
b-continuity of f at x_o implies that there is a b-open set L in (X,τ) containing x_o such that $f(L) \subseteq V$.
Since , $x_n \xrightarrow{b\text{-}cgt} x_o$, there exists a natural number m such that $n \geqslant m \Rightarrow x_n \in L \Rightarrow f\{x_n\} \in V$ .Combining these , we say that the sequence $\{f(x_n)\}$ is convergent to $f(x_o)$ because for every σ - open set V containing f (xo) ,there exists a natural number m such that
$n \geqslant m \Rightarrow f(x_n) \in V$.
Hence , symbolically ,
$x_n \xrightarrow{b\text{-}cgt} X_o \Rightarrow f(x_n) \xrightarrow{cgt} f(X_o)$, $\forall$ b-continuous maps f.

Hence, the theorem

Corollary (3.1):
If f: $(X,\tau) \rightarrow (Y,\sigma)$ be a b-irresolute mapping and $\{x_n\}$ be b- convergent to $x_o \in x$ then,
$x_n \xrightarrow{b\text{-}cgt} x_O \Rightarrow f(x_n) \xrightarrow{b\text{-}cgt} f(x_o)$

Proof:-
The proof is straight forward & natural , so omitted.
We , now, produce the following theorem concerned with b-continuous image of a sequentially b- compact set of a topological space.

Theorem (3.4):
A b-continuous image of a sequentially b – compact set is sequentially compact .

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Proof:

Suppose is a b-continuous mapping. let A be a sequentially b-compact set in topological space $(x,\tau)$and we have to show that f(A) is sequentially compact subset of (Y,σ) where $f\,(x,\tau) \rightarrow (Y,\sigma)$.

Let $\{y_n\}$ be an arbitrary sequence of points in f(A) , then for each $n \in N$ there exists $x_n \in A$ such that $f\,(x_n) = y_n$ and thus we obtain a sequence $\{x_n\}$ of points of A.

But A is sequentially b-compact w.r.t. τ so that there is a subsequence $\{x_{nk}\}$ of $\{x_n\}$ which is b-compact to a point say , x of A.

Therefore, $\{x_{nk}\} \xrightarrow{b\text{-cgt}} x \Rightarrow f\,(x_{nk}) \longrightarrow f(x) \in f(A)$

as f is b-continuous.

Hence, $f\,(x_{nk})$ is a subsequence of the sequence $\{y_n\}$ of f(A), converging to a point f (x) in f(A) . Consequently, f (A) is sequentially compact.

Corollary (3.2) : The b-irresolute image of a sequentially b-compact set is a sequentially b – compact set is a sequentially b – compact .

This means that sequentially b – compactness is a topological property under b – irresolute mappings.

CONCLUSION:

Since, compactness is one of the most important useful and fundamental concepts in topology so its structural properties as emphasized in the form of b – open sets, b – convergent sequences,

b – Lindel of spaces etc opens a new horizon in the world of Mathematics through this paper. The structures mentioned in the paper have wide application and it surely pleases the Mathematician if one of his abstract structures finds an application [8] & [9].

The future scope of study is to obtain results in respective → b – paracompactness

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